Information Theory and Channel Coding

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III. Channel capacity

- In this chapter, we study channel capacity and examine several implications of the capacity theorem of Shannon.
- In particular, we examine the fundamental limit of how much information can be transmitted over a channel given some key parameters.
- We present mathematical models of discrete and continuous channels and explore how these models can describe realistic channels.
- We introduce the concept of mutual information and its relation to entropy and the channel capacity of both discrete and continuous channels.

A. Discrete memoryless channels

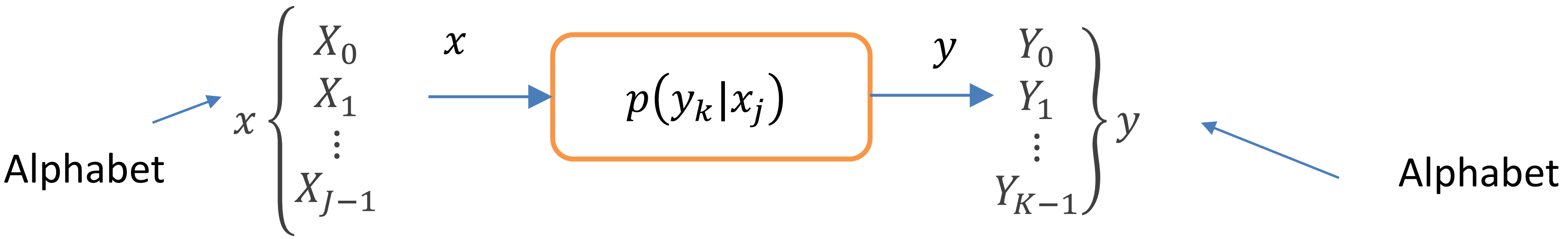
- Communication channels represent the medium over which signals are transmitted.

- In particular, communication channels introduce amplitude and phase distortions in the transmitted signals.

- Modelling communication channels is key because they can be simulated and their capacities can be computed.

- In this section, we will focus our attention on discrete memoryless channels using the concepts of random variables, probability and discrete memoryless sources.

- Let us consider a discrete memoryless channel (DMC) model as



Alphabet

$x \begin{cases} X_0 \\ X_1 \\ \vdots \\ X_{J-1} \end{cases}$

x

$p(y_k|x_j)$

y

$\left. \begin{matrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{K-1} \end{matrix} \right\} y$

Alphabet

- The model can be written as

$$y = x + n,$$

where n represents the noise.

- The model is discrete because y and x take on discrete values.

The mathematical description of discrete memoryless channels (DMCs) include:

- The input and output alphabets described by

$$x = \{X_0, X_1, \dots, X_{J-1}\} \text{ and } y = \{Y_0, Y_1, \dots, Y_{K-1}\}$$

- The set of transition probabilities given by

$$p(y_k|x_j) = \mathrm{P}(y_k = Y_k|x_j = X_j), \qquad \text{for all } j \text{ and } k$$

where $0 \leq p(y_k|x_j) \leq 1$ for all j and k.

- The channel can be completely characterized by the set of all transition probabilities as compactly described by

$$\boldsymbol{P} = \begin{bmatrix} p(y_0|x_0) & p(y_1|x_0) & \cdots & p(y_{K-1}|x_0) \\ p(y_0|x_1) & p(y_1|x_1) & \cdots & p(y_{K-1}|x_1) \\ \vdots & \vdots & \ddots & \vdots \\ p(y_0|x_{J-1}) & p(y_1|x_{J-1}) & \cdots & p(y_{K-1}|x_{J-1}) \end{bmatrix}$$

- A key property that applies to the set of transition probabilities is

$$\sum_{k=0}^{K-1} p(y_k|x_j) = 1, \qquad \text{for all } j$$

- The input x of the DMC is modelled by the probability

$$p(x_j) = P(x_j = X_j), \qquad j = 0,1, \dots, J-1$$

where $P(x_j = X_j)$ is the probability of an event.

- The joint probability mass function (pmf) of the input x and the ouput y of the DMC is described by

$$\begin{aligned} p(x_j, y_k) &= P(x_j = X_j, y_k = Y_k) \\ &= P(y_k = Y_k | x_j = X_j) P(x_j = X_j) \\ &= p(y_k | x_j) p(x_j) \end{aligned}$$

- The joint pmf is key as it contains the transition and input probabilities.

- The channel output is described by the pmf given by

$$
\begin{aligned}
p(y_k) &= P(y_k = Y_k) \\
&= \sum_{j=0}^{J-1} P\left(y_k = Y_k | x_j = X_j\right) P\left(x_j = X_j\right) \\
&= \sum_{j=0}^{J-1} p\left(y_k | x_j\right) p\left(x_j\right),\ k = 0,1,\dots,K-1
\end{aligned}
$$

- With the mathematical quantities that constitute the structure of DMCs it is possible to fully characterize them.

Example 1

Consider a binary symmetric channel with $J = K = 2$.

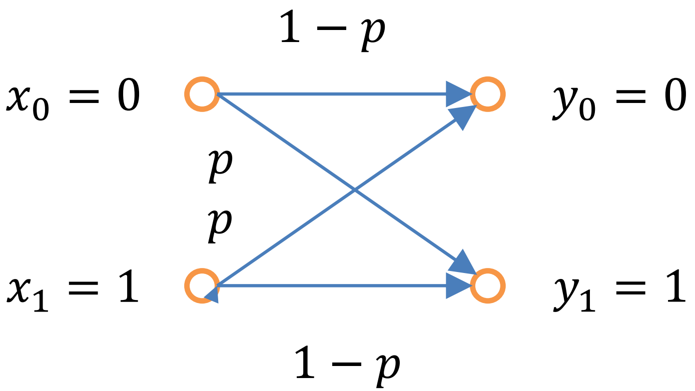
Since the channel is symmetric the probability of receiving a 1 if a 0 was sent is the same as the probability of receiving a 0 if a 1 was sent. This is known as the conditional probability of error and given by p.

a) Describe in a diagram the binary symmetric channel and all its probabilities.

b) Compute the input, transition and output probabilities.

a) The binary symmetric channel (BSC) of this problem deals with $J = 2$ inputs, namely, $x_0 = 0$ and $x_1 = 1$.

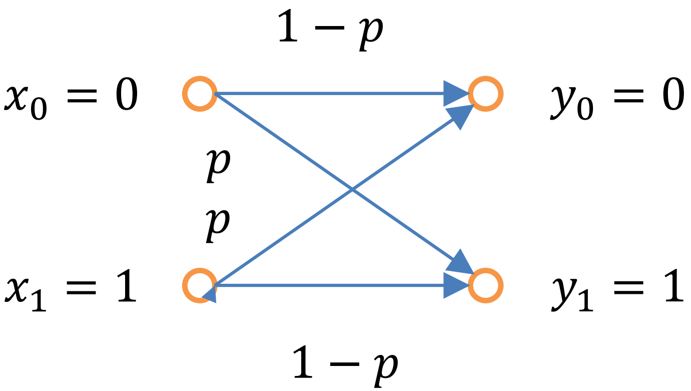
There are also $K = 2$ outputs, namely, $y_0 = 0$ and $y_1 = 1$. The BSC can then be illustrated by



b) The input probabilities are described by

$p(x_0) = P(x_0 = 0)$

$p(x_1) = P(x_1 = 1)$



$1-p$

$x_0 = 0$

$y_0 = 0$

p

p

$x_1 = 1$

$y_1 = 1$

$1-p$

The transition probabilities are given by

$p(y_0|x_0) = 1 - p$

$p(y_1|x_1) = 1 - p$

$p(y_1|x_0) = p$

$p(y_0|x_1) = p$

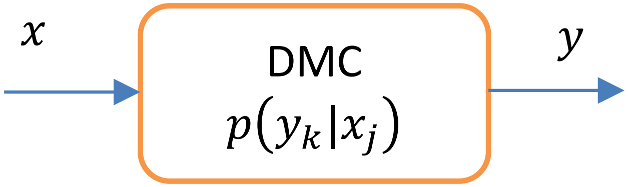
The output probabilities are described by

$$p(y_0) = \sum_{j=0}^{J-1} p(y_0|x_j)p(x_j) = p(y_0|x_0)p(x_0) + p(y_0|x_1)p(x_1) = (1-p)p(x_0) + pp(x_1)$$

$$p(y_1) = \sum_{j=0}^{J-1} p(y_1|x_j)p(x_j) = p(y_1|x_0)p(x_0) + p(y_1|x_1)p(x_1) = pp(x_0) + (1-p)p(x_1)$$

B. Mutual information

- Let us consider a DMC and the entropy associated with the input alphabet $H(x)$, which measures the uncertainty about the input x.



- An important question for DMCs is: how to measure $H(x)$ when observing y ?

- We can investigate this by looking into the concept of conditional entropy.

- The conditional entropy for a given output Y_k is described by

$$H(x|y_k = Y_k) = \sum_{j=0}^{J-1} p(x_j|y_k) \log_2 \left[\frac{1}{p(x_j|y_k)}\right]$$

- If we compute the mean value of $H(x|y_k = Y_k)$ then we obtain the conditional entropy

$$\begin{aligned} H(x|y) &= \sum_{k=0}^{K-1} H(x|y_k = Y_k) p(y_k) \\ &= \sum_{k=0}^{K-1} \sum_{j=0}^{J-1} p(x_j|y_k) p(y_k) \log_2 \left[\frac{1}{p(x_j|y_k)}\right] \\ &= \sum_{k=0}^{K-1} \sum_{j=0}^{J-1} p(x_j, y_k) \log_2 \left[\frac{1}{p(x_j|y_k)}\right] \end{aligned}$$

- The conditional entropy $H(x|y)$ measures the uncertainty of the channel after observing the ouput y.

- The mutual information measures the uncertainty about the input x of the DMC while observing the output y of the DMC.

- The mutual information is described by

$$I(x,y) = H(x) - H(x|y),$$

where $H(x)$ measures the uncertainy of the input x and $H(x|y)$ measures the uncertainty of the DMC after observing the ouput y of the DMC.

- There is an equivalence of the mutual information if we swap the input and the ouput of the DMC, which yields

$$I(y,x) = H(y) - H(y|x),$$

Properties

i) The mutual information $I(x,y)$ is symmetric, i.e.,

$$I(x,y) = I(y,x)$$

ii) The mutual information is always nonnegative, i.e.,

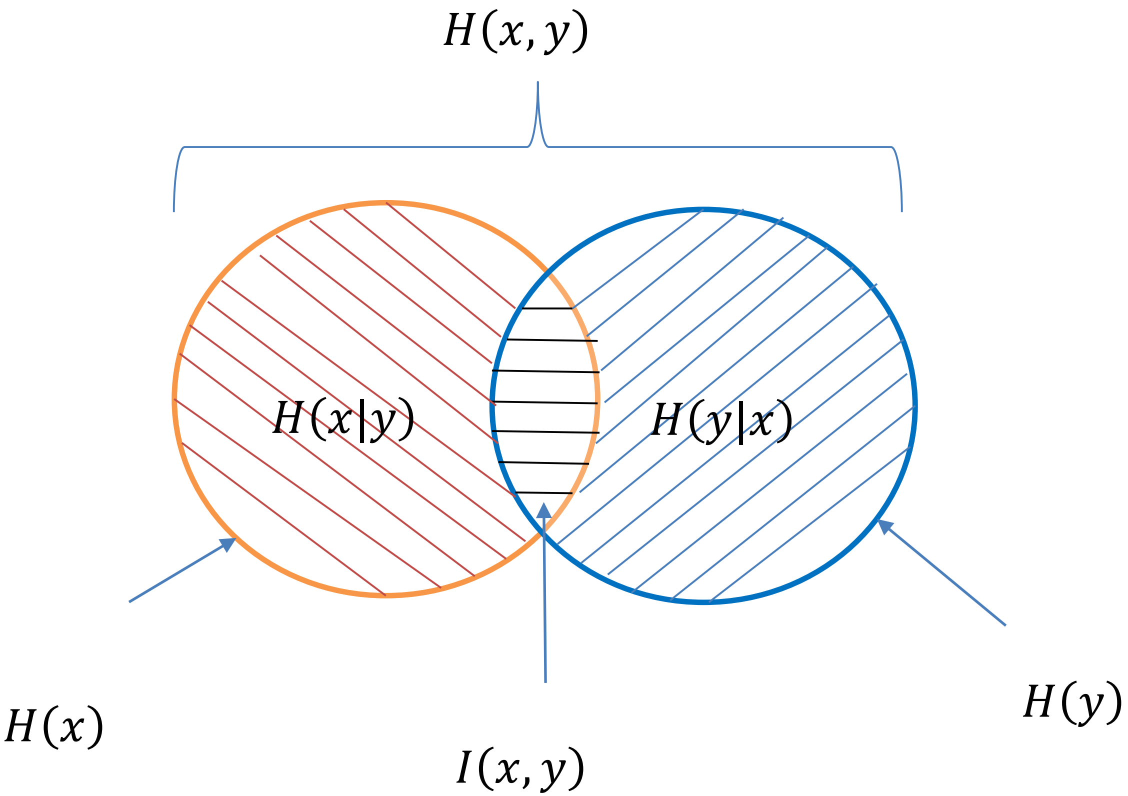
$$I(x,y) \geq 0$$

iii) The mutual information $I(x,y)$ is related to the joint entropy of the input and the output of the channel by

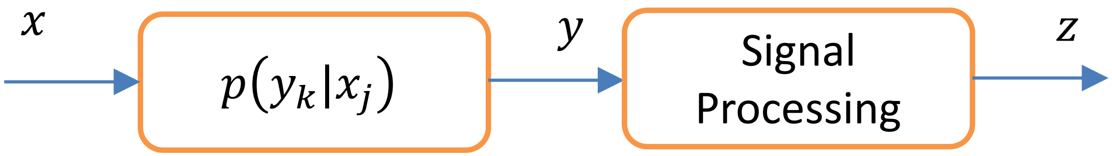
$$I(x,y) = H(x) + H(y) - H(x,y),$$

$$H(x,y) = \sum_{k=0}^{K-1} \sum_{j=0}^{J-1} p(x_j, y_k) \log_2 \left[\frac{1}{p(x_j, y_k)} \right]$$

Illustration



iv) Data/information processing inequality



$$I(x,y) \geq I(x,z)$$

No clever signal processing can increase the information content.

Proof of property i)

We first use the formula for entropy and further manipulate it as follows:

$$\begin{aligned}H(x) &= \sum_{j=0}^{J-1} p(x_j)\log_2\left[\frac{1}{p(x_j)}\right]\\ &= \sum_{j=0}^{J-1} p(x_j)\log_2\left[\frac{1}{p(x_j)}\right]\sum_{k=0}^{K-1} p(y_k|x_j)\\ &= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(y_k|x_j)\,p(x_j)\log_2\left[\frac{1}{p(x_j)}\right]\\ &= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(y_k,x_j)\log_2\left[\frac{1}{p(x_j)}\right]\end{aligned}$$

Substituting $H(x)$ and $H(x|y)$ into $I(x,y)$, we obtain

$$\begin{aligned}I(x,y) &= H(x) - H(x|y)\\ &= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(y_k,x_j)\log_2\left[\frac{p(x_j|y_k)}{p(x_j)}\right]\end{aligned}$$

Using Bayes' rule for conditional probabilities, we have

$$\frac{p(x_j|y_k)}{p(x_j)} = \frac{p(y_k|x_j)}{p(y_k)}$$

Substituting the above relation into $I(x, y)$, we obtain

$$\begin{aligned} I(x,y) &= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(y_k, x_j)\log_2\left[\frac{p(x_j|y_k)}{p(x_j)}\right] \\ &= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(y_k, x_j)\log_2\left[\frac{p(y_k|x_j)}{p(y_k)}\right] \\ &= I(y, x\ ) \end{aligned}$$

Proof of property ii)

Since $p(x_j|y_k) = \frac{p(y_k, x_j)}{p(y_k)}$, we may express the mutual information of the channel as

$$I(x,y) = \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} p(y_k, x_j) \log_2 \left[\frac{p(x_j|y_k)}{p(x_j)}\right]$$
$$= \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} p(y_k, x_j) \log_2 \left[\frac{p(y_k, x_j)}{p(y_k)p(x_j)}\right]$$

Using the fundamental inequality arising from Jensen's inequality $\sum_{k=0}^{K-1} p_k \log_2 \left[\frac{q_k}{p_k}\right] \leq 0$, we obtain

$$I(x,y) \geq 0$$

The equality only holds if

$$p(y_k, x_j) = p(y_k)p(x_j)$$

and then we have

$$I(x, y) = 0$$

This property shows that we cannot lose information on average by observing the output of a channel.

Moreover, the mutual information is zero only if the random variables x and y are statistically independent.

Proof of property iii)

Let us first rewrite the expression of the joint entropy $H(x,y)$ as

$$
\begin{aligned}
H(x,y) &= \sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j,y_k)\log_2\left[\frac{1}{p(x_j,y_k)}\right] \\
&= \sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j,y_k)\log_2\left[\frac{p(x_j)p(y_k)}{p(x_j,y_k)}\right] \\
&+ \sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j,y_k)\log_2\left[\frac{1}{p(x_j)p(y_k)}\right]
\end{aligned}
$$

The first double summation on the right-hand side of the above expression is the negative of the mutual information, i.e., $-I(x,y)$.

The second term can be manipulated as follows:

$$\sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j, y_k)\log_2\left[\frac{1}{p(x_j)p(y_k)}\right] =$$

$$= \sum_{j=0}^{J-1}\log_2\left[\frac{1}{p(x_j)}\right]\sum_{k=0}^{K-1} p(x_j, y_k) + \sum_{k=0}^{K-1}\log_2\left[\frac{1}{p(y_k)}\right]\sum_{j=0}^{J-1} p(x_j, y_k)$$

$$= \sum_{j=0}^{J-1} p(x_j)\log_2\left[\frac{1}{p(x_j)}\right] + \sum_{k=0}^{K-1} p(y_k)\log_2\left[\frac{1}{p(y_k)}\right]$$

$$= H(x) + H(y)$$

Accordingly, we have

$$H(x, y) = -I(x, y) + H(x) + H(y)$$

and

$$I(x, y) = H(x) + H(y) - H(x, y)$$

Proof of property iv)

The data processing inequality can be used to show that no clever manipulation of the data (or signals) can increase the information content.

Let us employ the chain rule with the mutual information to write

$$\begin{aligned} I(x,y,z) &= I(x,z) + I(x,y|z) \\ &= I(x,y) + I(x,z|y) \end{aligned}$$

Since x and z are conditionally independent given y, we have

$$I(x,z|y) = 0$$

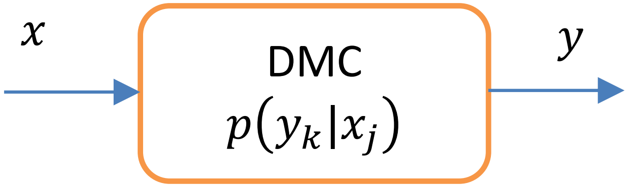
Since $I(x,y|z) \geq 0$, we have

$$I(x,y) \geq I(x,z)$$

The equality only holds when x, y and z form a Markov chain.

C. Capacity of discrete memoryless channels

- Let us consider a DMC and the entropy associated with the input alphabet $H(x)$, which measures the uncertainty about the input x.



- The mutual information of the input x and the output y of the channel is given by

$$I(x,y) = \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(y_k, x_j)\log_2\left[\frac{p(x_j|y_k)}{p(x_j)}\right]$$

$$= \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(y_k, x_j)\log_2\left[\frac{p(y_k|x_j)}{p(y_k)}\right]$$

- The joint pmf between the input and output variables is given by

$$p(y_k, x_j) = p(y_k|x_j)p(x_j)$$

- The output probabilities can be computed by

$$p(y_k) = \sum_{j=0}^{J-1} p(y_k|x_j)p(x_j), \qquad k = 0,1, \dots, K-1$$

- In order to compute $I(x, y)$, we need the input probabilities

$$p(x_j), \qquad j = 0,1, \dots, J-1$$

- The capacity of a DMC can be computed by maximizing the mutual information $I(x, y)$ subject to appropriate constraints on $p\left(x_j\right)$.

- The computation of the capacity can be formulated as the optimization:

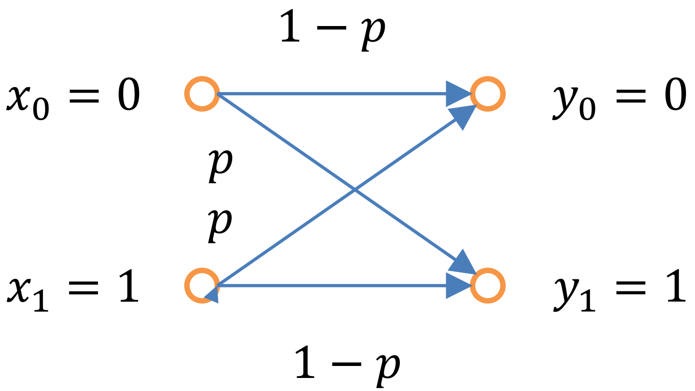
$$C = \max_{p(x_j)} I(x, y) \text{ bits/channel use or bits / transmission}$$

$$\text{subject to } p\left(x_j\right), \quad \text{for all } j \text{ and } \sum_{j=0}^{J-1} p\left(x_j\right) = 1$$

- The optimization involves the maximization of $I(x, y)$ by adjusting the variables $p(x_1), p(x_2), \dots, p\left(x_{J-1}\right)$ subject to appropriate constraints.

Example 2

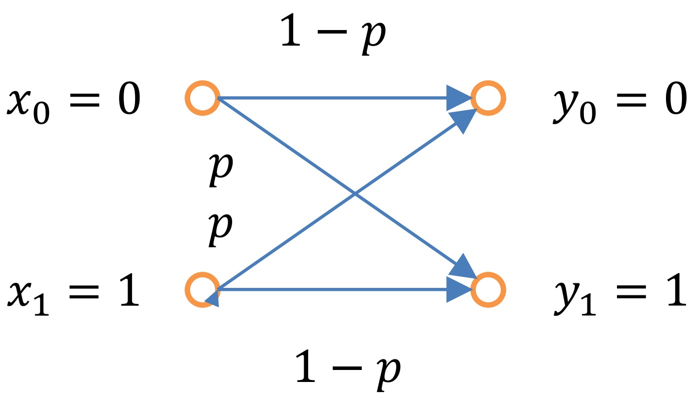
Consider the BSC illustrated by



a) Compute the capacity of the channel

b) Show how the capacity varies with p using a plot.

We consider the BSC.



We know that the entropy $H(x)$ is maximized when $p(x_0) = p(x_1) = \frac{1}{2}$, where x_0 and x_1 are 0 and 1, respectively.

The mutual information $I(x, y)$ is similarly maximized as described by

$$C = I(x, y) \text{ when } p(x_0) = p(x_1) = \frac{1}{2},$$

where

$$p(y_0|x_0) = 1 - p = p(y_1|x_1)$$
$$p(y_1|x_0) = p = p(y_0|x_1)$$

a) By substituting the transition probabilities in $I(x,y)$, we obtain

$$I(x,y) = \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p\left(y_k, x_j\right)\log_2\left[\frac{p(y_k|x_j)}{p(y_k)}\right]$$

With $J = K = 2$ and then setting $p(x_0) = p(x_1) = \frac{1}{2}$, we have

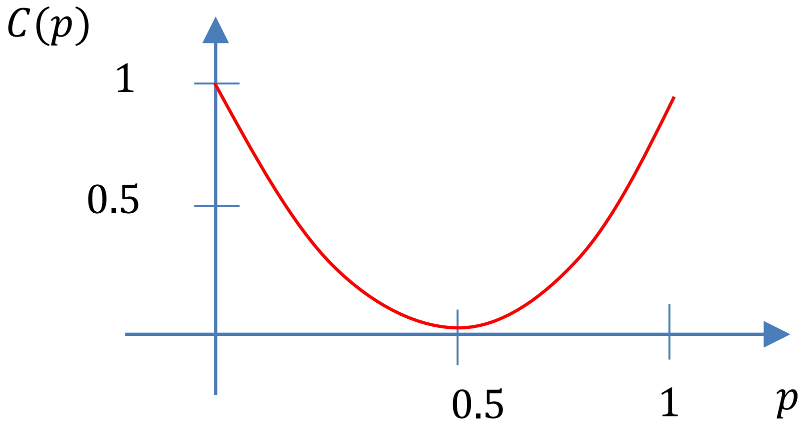
$$\begin{aligned}
C &= \max_{p(x_j)}\sum_{j=0}^{1}\sum_{k=0}^{1} p\left(y_k, x_j\right)\log_2\left[\frac{p(y_k|x_j)}{p(y_k)}\right] \\
&= p(y_0,x_0)\log_2\left[\frac{p(y_0|x_0)}{p(y_0)}\right] + p(y_0,x_1)\log_2\left[\frac{p(y_0|x_1)}{p(y_0)}\right] \\
&\quad + p(y_1,x_0)\log_2\left[\frac{p(y_1|x_0)}{p(y_1)}\right] + p(y_1,x_1)\log_2\left[\frac{p(y_1|x_1)}{p(y_1)}\right] \\
&= p(y_0|x_0)\,p(x_0)\log_2\left[\frac{p(y_0|x_0)}{p(y_0)}\right] + p(y_0|x_1)p(x_1)\log_2\left[\frac{p(y_0|x_1)}{p(y_0)}\right] \\
&\quad + p(y_1|x_0)p(x_0)\log_2\left[\frac{p(y_1|x_0)}{p(y_1)}\right] + p(y_1|x_1)p(x_1)\log_2\left[\frac{p(y_1|x_1)}{p(y_1)}\right] \\
&= \frac{1-p}{2}\log_2[2(1-p)] + \frac{p}{2}\log_2[2p] + \frac{p}{2}\log_2[2p] + \frac{1-p}{2}\log_2[2(1-p)] \\
&= 1 + \mathrm{p}\log_2 p + (1-p)\log_2(1-p)
\end{aligned}$$

b) Using the definition of entropy and their mathematical relations we have the capacity of the BSC

$$C(p) = 1 - H(p),$$

where $\mathrm{H(p)} = -\mathrm{plog}_2\, p - (1-p)\log_2(1-p)$.

The channel capacity varies with p in a convex manner as shown below.



When $p = 0$, C attains its maximum value of 1 bit/ channel use

When $p = \frac{1}{2}$, C attains its minimum value of 0 bit/ channel use (useless channel)

D. Differential entropy and mutual information for continuous variables

- In this section, we extend the previous concepts to continuous sources and channels, which are modelled as continuous random variables.

- Consider a random variable x with the probability density function $p_x(X)$, the differential entropy of x is described by

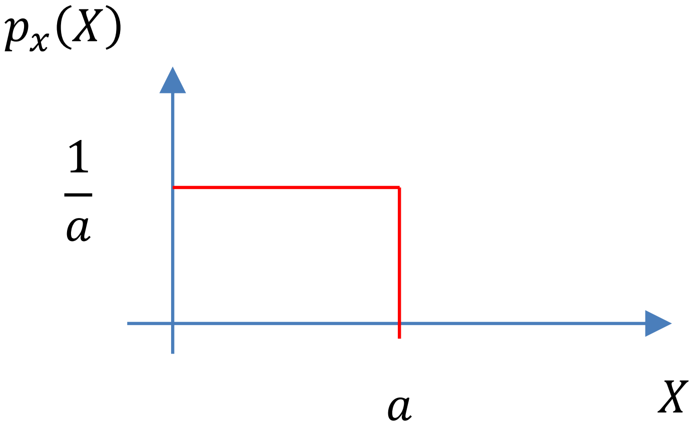
$$h(x) = \int_{-\infty}^{\infty} p_x(X) \log_2 \left[\frac{1}{p_x(X)}\right] dX$$

- As in the discrete case, the differential entropy depends only on the probability density of the random variable x.

Example 3

Compute the differential entropy of a random variable with uniform distribution described by

$$p_x(X) = \begin{cases} \dfrac{1}{a}, & 0 < X < a \\ 0, & \text{otherwise} \end{cases}$$



Solution:

$$h(x) = \int_{-\infty}^{\infty} p_x(X) \log_2 \left[\frac{1}{p_x(X)}\right] dX$$

$$= \int_0^a \frac{1}{a} \log_2 a \, dX$$

$$= \log_2 a \text{ bits}$$

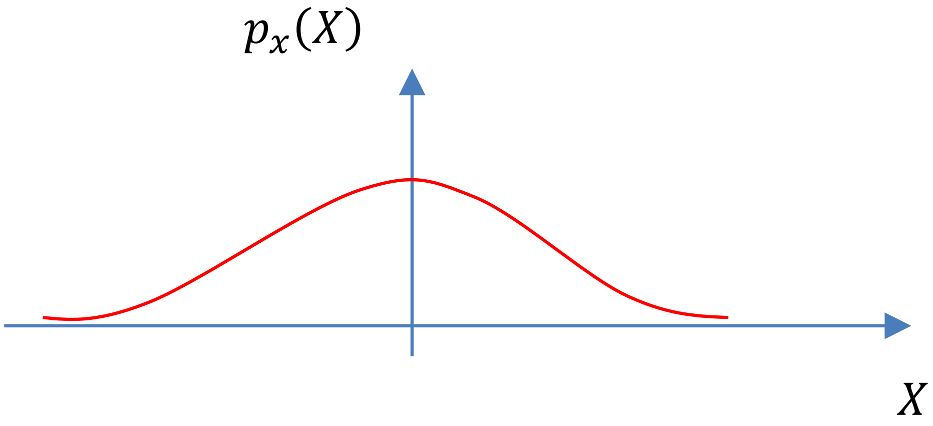
Note that $\log_2 a < 0$ for $a < 1$.

The entropy of a continuous random variable can be negative unlike the case for a discrete random variable.

Example 4

Compute the differential entropy of a random variable with Gaussian distribution described by

$$p_x(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{X^2}{2\sigma^2}}$$



Solution:

$$\mathrm{h}(x) = \int_{-\infty}^{\infty} p_x(X) \ln\left[\frac{1}{p_x(X)}\right] dX \quad \text{(nats)}$$

$$= -\int_{-\infty}^{\infty} p_x(X) \ln p_x(X)\, dX$$

$$= -\int_{-\infty}^{\infty} p_x(X) \left[-\frac{X^2}{2\sigma^2} - \ln\sqrt{2\pi\sigma^2}\right] dX$$

$$= \frac{1}{2}\ln 2\pi\sigma^2 + \frac{1}{2}\frac{E\left[x^2\right]}{\sigma^2}$$

$$= \frac{1}{2}\ln 2\pi\sigma^2 + \frac{1}{2}\ln \mathrm{e}$$

$$= \frac{1}{2}\ln 2\pi e\sigma^2 \text{ nats}$$

Changing the basis from $\ln$ to $\log_2$, we have

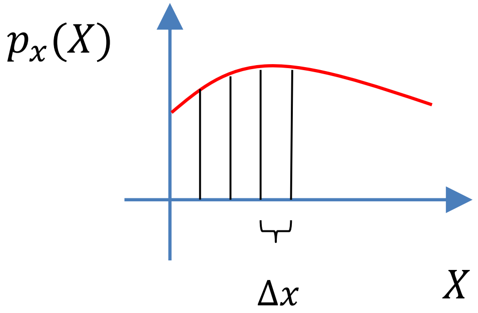
$$\mathrm{h}(x) = \frac{1}{2}\log_2 2\pi e\sigma^2 \quad \text{bits}$$

Relation of differential entropy to entropy of discrete variables

- Let us consider the random variable x as the limiting form of a discrete random variable $x_k = k\Delta x, k = 0, \pm 1, \pm 2, \ldots$, where $\Delta x \rightarrow 0$.

- In this case, x takes on a value in the range $[x_k, x_k + \Delta x\ ]$ with probability given by

$$p_x(X_k)\Delta x = \int_{k\Delta x}^{(k+1)\Delta x} p_x(X)dX$$



- Consider the quantized random variable x_q described by

$$x_q = x_k, \qquad k\Delta x \leq X_q < (k+1)\Delta x$$

- Then the probability that $x_q = X_k$ is given by

$$P(x_q = X_k) = p_x(X_k)\Delta x = \int_{k\Delta x}^{(k+1)\Delta x} p_x(X)dX$$

- Let us now compute the entropy of x_k by letting $\Delta x \to 0$ as follows:

$$\begin{aligned}
H(x_k) &= \lim_{\Delta x \to 0} \sum_{k=-\infty}^{\infty} p_x(X_k)\,\Delta x \log_2\left(\frac{1}{p_x(X_k)\Delta x}\right) \\
&= \lim_{\Delta x \to 0} \left[\sum_{k=-\infty}^{\infty} p_x(X_k)\,\Delta x \log_2\left(\frac{1}{p_x(X_k)}\right) - \log_2 \Delta x \sum_{k=-\infty}^{\infty} p_x(X_k)\Delta x\right] \\
&= \int_{-\infty}^{\infty} p_x\,(X) \log_2\left(\frac{1}{p_x(X)}\right) dX - \lim_{\Delta x \to 0} \log_2 \Delta x \int_{-\infty}^{\infty} p_x\,(X)\mathrm{dX} \\
&= h(x) - \lim_{\Delta x \to 0} \log_2 \Delta x
\end{aligned}$$

Theorem 1:

The previous development leads to

$$H(x_k) = h(x) - \lim_{\Delta x \to 0} \log_2 \Delta x$$

or

$$h(x) = H(x_k) + \lim_{\Delta x \to 0} \log_2 \Delta x,$$

which for $\Delta x \to 0$ results in

$$h(x) = H(x_k)$$

and for an arbitrary Δx related to n quantization bits yields

$$h(x) = H(x_k) + \log_2 \Delta x = H(x_k) + n$$

Example 5

Compute the entropy for the following cases:

a) If a random variable x *has* uniform distribution on $[0,1]$ and we let $\Delta x{=}2^{-n}$.

b) If a random variable x *has* Gaussian distribution with zero mean, $\sigma^2 = 100$.

Solution:

a) For a random variable x with uniform distribution on $[0, 1]$ and $\Delta x{=}2^{-n}$, we have

$$H(x_k) = \sum_{k=-\infty}^{\infty} p_x(X_k)\,\Delta x\,\log_2\left(\frac{1}{p_x(X_k)\Delta x}\right) = n$$

and

$$h(x) = H(x_k) + \log_2 \Delta x = n - n = 0,$$

which means that n bits suffice to describe x to an accuracy of n bits.

b)

For a random variable x *with* Gaussian distribution with zero mean and $\sigma^2 = 100$, we have

$$h(x) = H(x_k) + \log_2 \Delta x \ = H(x_k) + n$$
$$= \frac{1}{2} \log_2 2\pi e \sigma^2 + \mathrm{n} = 5.37 \mathrm{bits} + \mathrm{n}$$

Joint and conditional entropy: extension to vectors

- We can extend the definition of differential entropy to random vectors.

- The joint differential entropy for a random vector $\boldsymbol{x} = [x_1 \quad \cdots \quad x_n]^T$ is defined by

$$h(\boldsymbol{x}) = \int_{-\infty}^{\infty} p_{\boldsymbol{x}}(\boldsymbol{X}) \log_2 \left[\frac{1}{p_{\boldsymbol{x}}(\boldsymbol{X})}\right] d\boldsymbol{X}$$

- The conditional differential entropy of two variables x and y is described by

$$h(x|y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{x,y}(X,Y) \log_2 \left[\frac{1}{p_x(X|Y)}\right] dXdY$$

- Since in general $p_x(X|Y) = p_{x,y}(X,Y)/p_y(Y)$, we can write

$$h(x|y) = h(x,y) - h(y)$$

Example 6

Compute the differential entropy of the random vector $\boldsymbol{x} = [x_1 \quad \cdots \quad x_n]^T$ whose joint probability density function is

$$p_{\boldsymbol{x}}(\boldsymbol{X}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\boldsymbol{K})}} e^{-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{m}_x)^T \boldsymbol{K}^{-1}(\boldsymbol{X}-\boldsymbol{m}_x)}$$

Solution:

$$h(\boldsymbol{x}) = \int_{-\infty}^{\infty} p_{\boldsymbol{x}}(\boldsymbol{X})\, ln\left[\frac{1}{p_{x}(X)}\right] d\boldsymbol{X} \quad \text{(nats)}$$

$$= -\int_{-\infty}^{\infty} p_{\boldsymbol{x}}(\boldsymbol{X}) \left(-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{m}_x)^T \boldsymbol{K}^{-1}(\boldsymbol{X}-\boldsymbol{m}_x) - \ln(2\pi)^{\frac{n}{2}} \det(\boldsymbol{K})^{\frac{1}{2}}\right) d\boldsymbol{X}$$

$$= \frac{1}{2}\mathrm{E}[(\boldsymbol{x}-\boldsymbol{m}_x)^T \boldsymbol{K}^{-1}(\boldsymbol{x}-\boldsymbol{m}_x)] + \frac{1}{2}\ln(2\pi)^n \det(\boldsymbol{K})$$

$$= \frac{1}{2} tr[\boldsymbol{K}\boldsymbol{K}^{-1}] + \frac{1}{2}\ln(2\pi)^n \det(\boldsymbol{K})$$

$$= \frac{1}{2} n \ln e \; + \frac{1}{2}\ln(2\pi)^n \det(\boldsymbol{K})$$

$$= \frac{1}{2} \ln e^n \; + \frac{1}{2}\ln(2\pi)^n \det(\boldsymbol{K})$$

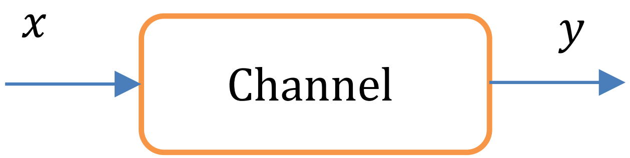
$$= \frac{1}{2}\ln(2\pi e)^n \det(\boldsymbol{K})$$

By changing the basis of the logarithm, we have

$$h(\boldsymbol{x}) = \frac{1}{2}\log_2(2\pi e)^n \det(\boldsymbol{K}) \;\; \text{bits}$$

E. Mutual information

- Consider a pair of random variables x and y that can represent the input and the output of a communication channel.



- The mutual information between x and y is defined by

$$I(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{x,y}(X,Y)\log_2\left[\frac{p_x(X|Y)}{p_x(X)}\right] dXdY\,,$$

where $p_{x,y}(X,Y)$ is the joint pdf of x and y , and $p_x(X|Y)$ is the conditional pdf of x subject to $y = Y$.

- The conditional differential entropy of two variables x and y is described by

$$h(x|y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{x,y}(X,Y)\log_2\left[\frac{1}{p_x(X|Y)}\right] dXdY$$

- Since in general $p_x(X|Y) = p_{x,y}(X,Y)/p_y(Y)$, we can write

$$h(x|y) = h(x,y) - h(y)$$

- The mutual information is then given by

$$I(x,y) = h(x) - h(x|y)$$

- These relations are useful to compute the mutual information in practical situations.

Properties of mutual information

i) $I(x, y) = I(y, x)$ (symmetry)

ii) $I(x, y) \geq 0$ (non negativity)

iii) $I(x, y) = h(x) - h(x|y)$

$$= h(y) - h(y|x)$$

iv) Data processing inequality: $I(x, y) \geq I(x, z)$

- The proofs are similar to those of mutual information with discrete variables.

Example 7

Compute the mutual information between the input x and the output y of the channel

when both x and y are drawn from Gaussian random variables with zero mean and variance σ^2 and the covariance matrix of $\boldsymbol{u} = [x\ y]^{\boldsymbol{T}}$

$$\boldsymbol{K} = E[(\boldsymbol{u} - \boldsymbol{m}_u)(\boldsymbol{u} - \boldsymbol{m}_u)^T] = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix},$$

where $\boldsymbol{m}_u$ is the mean vector of $\boldsymbol{u}$.

Solution:

The differential entropies of the input x and the output y of the channel are

$$h(x) = \frac{1}{2}\log_2(2\pi e)\sigma^2 = h(y)$$

The joint differential entropy is given by

$$\begin{aligned} h(x, y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{x,y}(X, Y)\log_2\left[\frac{1}{p_{x,y}(X,Y)}\right] dXdY \\ &= \frac{1}{2}\log_2(2\pi e)^2 \det(\boldsymbol{K}) \\ &= \frac{1}{2}\log_2(2\pi e)^2\sigma^4(1-\rho^2) \end{aligned}$$

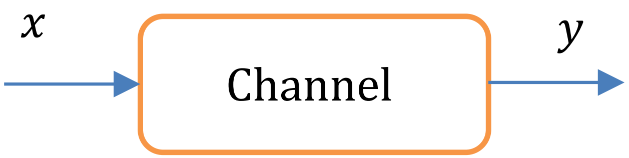
Therefore, the mutual information is described by

$$\begin{aligned} I(x,y) &= h(x) - h(x|y) \\ &= h(x) + h(y) - h(x,y) \\ &= \frac{1}{2}\log_2(2\pi e)\sigma^2 + \frac{1}{2}\log_2(2\pi e)\sigma^2 - \frac{1}{2}\log_2(2\pi e)^2\sigma^4(1-\rho^2) \\ &= -\frac{1}{2}\log_2(1-\rho^2), \end{aligned}$$

where $h(x|y) = h(x,y) - h(y)$

F. Capacity of Gaussian channels

- The information capacity of Gaussian channels is the maximum of the mutual information between the input and the output of the channel.



- To this end, we need to consider all distributions on the input that satisfy a power constraint P.

- Mathematically, the information capacity of Gaussian channels with power constraint P is given by

$$C = \max_{p_x(X)} I(x, y)$$

$$\text{subject to } E[x^2] \leq P$$

Channel capacity theorem (Shannon, 1948)

The information capacity of a continuous channel bandlimited to B Hz perturbed by additive white Gaussian noise (AWGN) with power spectral density $\frac{N_0}{2}$ is given by

$$C = B\log_2\left(1+\frac{P}{N_0 B}\right), \quad \text{bits/ s}$$

where P is average transmit power.

This theorem shows that given P and B we can transmit information at a rate of C bits per second.

Computation of the information capacity

- In order to solve the optimization problem given by

$$C = \max_{p_x(X)} I(x,y)$$

$$\text{subject to } E[x^2] \leq P$$

- We first consider the channel model described by

x
AWGN channel
y

$$y = x + n,$$

where n is AWGN with zero mean and variance σ^2.

- We then work out the mutual information expression as follows:

$$I(x,y) = h(y) - h(y|x)$$

- The mutual information expression can be simplified as

$$\begin{aligned} I(x,y) &= h(y) - h(y|x) \\ &= h(y) - h(x+n|x) \\ &= h(y) - h(n|x) \\ &= h(y) - h(n), \end{aligned}$$

which takes into account that x and n are statistically independent.

- Next, we need to compute the differential entropies $h(y)$ and $h(n)$.

- The differential entropy of AWGN is given by

$$h(n) = \frac{1}{2}\log_2(2\pi e\sigma^2)$$

- Now, we need to compute the variance of y, which is given by

$$\sigma_y^2 = E[y^2]$$
$$= E[(x+n)^2] = E[x^2] + E[n^2] = P + \sigma^2$$

- The differential entropy of y is expressed by

$$h(y) = \frac{1}{2}\log_2\left(2\pi e \sigma_y^2\right)$$
$$= \frac{1}{2}\log_2\left(2\pi e(P+\sigma^2)\right)$$

- The capacity is the maximum of the mutual information subject to the power constraint, which is taken into account in $h(y)$, and yields

$$\begin{aligned} \mathrm{C_t} &= \max I(x,y) = h(y) - h(n) \\ &= \frac{1}{2}\log_2\big(2\pi e(P+\sigma^2)\big) - \frac{1}{2}\log_2(2\pi e\sigma^2) \\ &= \frac{1}{2}\log_2\left(\frac{P+\sigma^2}{\sigma^2}\right) \\ &= \frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \text{ bits / transmission} \end{aligned}$$

- We note that the maximization of $h(y)$ requires that y be Gaussian as Gaussian random variables have the largest differential entropy.

- The capacity can also be expressed per unit of time by considering that K samples have been transmitted over T seconds, which results in

$$\begin{aligned} \mathrm{C} &= \frac{\mathrm{K}}{\mathrm{T}}\mathrm{C_t} = \frac{\mathrm{K}}{\mathrm{T}}\frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \\ &= \frac{2\mathrm{BT}}{\mathrm{T}}\frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \\ &= B\log_2\left(1+\frac{P}{N_0 B}\right) \text{ bits / second} \end{aligned}$$

- In the above expression, which has been derived by Shannon, we make use of $\mathrm{K} = 2\mathrm{BT}$ samples, where B is the bandwidth.

G. Implications of the channel capacity theorem

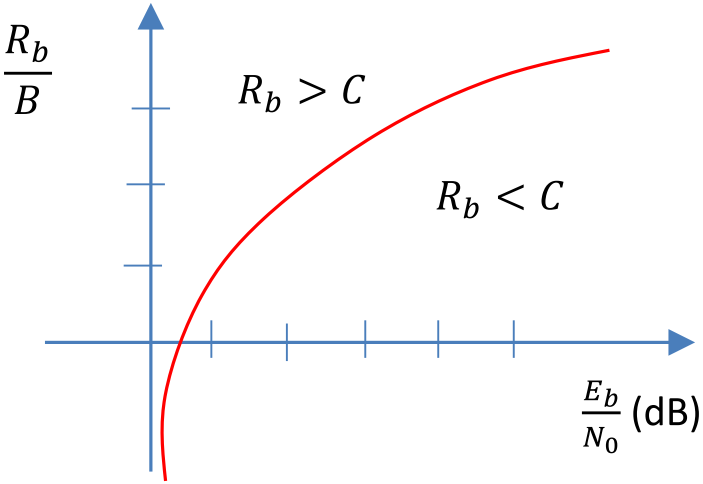
- In an ideal system, we transmit at a rate equal to $R_b = C$ bits /s.

- If we take into account $P = E_b C$, where E_b is the transmit energy per bit, we have

$$\frac{C}{B} = \log_2\left(1 + \frac{P}{N_0 B}\right) = \log_2\left(1 + \frac{E_b C}{N_0 B}\right)$$

- The spectral efficiency is the ratio of energy per bit by power spectral density is given by

$$\frac{E_b}{N_0} = \frac{2^{\frac{C}{B}} - 1}{\frac{C}{B}}$$



i) When $B \rightarrow \infty \frac{E_b}{N_0}$ approaches

$$\left(\frac{E_b}{N_0}\right)_\infty = \lim_{B \rightarrow \infty} \left(\frac{E_b}{N_0}\right)$$

$$= \frac{1}{\log_2 e} = 0.693 \text{ or } -1.6 \text{ dB}$$

Shannon limit

The capacity limit is then given by

$$C_\infty = \lim_{B \rightarrow \infty} \mathrm{C} = \frac{\mathrm{P}}{\mathrm{N}_0} \log_2 \mathrm{e}$$

Proof

Since $\log_2(1+x) = x\, log_2\left((1+x)^{\frac{1}{x}}\right)$ and $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$, we have

$$\frac{C}{B} = \log_2\left(1 + \frac{P}{N_0 B}\right)$$

$$= \frac{C}{B}\frac{E_b}{N_0}\log_2\left(1 + \frac{C}{B}\frac{E_b}{N_0}\right)^{\frac{N_0 B}{C E_b}}$$

We can then simplify the above as

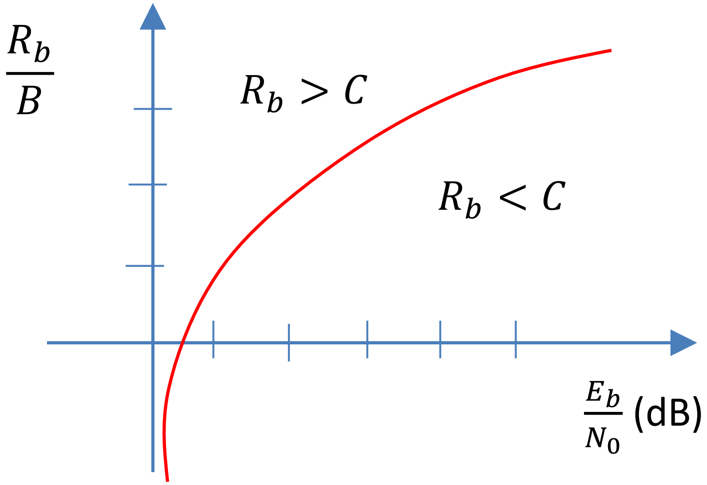
$$\frac{E_b}{N_0}\log_2\left(1 + \frac{C}{B}\frac{E_b}{N_0}\right)^{\frac{N_0 B}{C E_b}} = 1$$

If $\frac{C}{B} \to 0$ or $B \to\ \infty$ then we obtain

$$\frac{E_b}{N_0} = \frac{1}{\log_2 \mathrm{e}} = 0.693$$

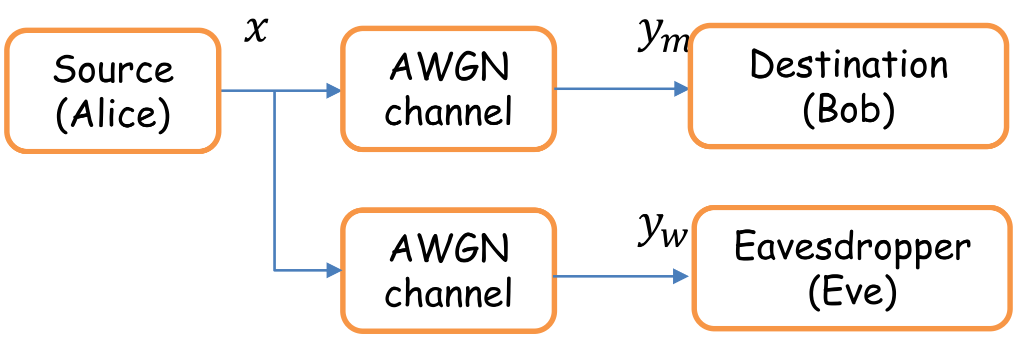
ii) Capacity bound $R_b = C$

- When $R_b \leq C \rightarrow$ error-free transmission is possible
- When $R_b > C \rightarrow$ error-free transmission is not possible



H. Capacity and Security

- The notion of capacity can be extended to physical layer security, which is illustrated by the diagram below.



- The secrecy capacity refers to the maximum information rate at which an eavesdropper cannot decode information transmitted by a source.

- The secrecy capacity is defined as the difference between the capacity of the main and the wiretap channels:

$$C_s = C_m - C_w$$

C.E. Shannon, "Communication theory of secrecy systems," *Bell Syst. Tech. Journ.*, vol. 29, pp. 656–715, 1949.
S. K. Leung-Yan-Cheong and M. E. Hellman, "The Gaussian wiretap channel," IEEE Trans. on Inform. Theory, vol. 24, no. 4, pp. 451456, July 1978.

- Let us now compute the secrecy capacity of the system as follows.

- For the main and wiretap channels we solve the optimization problem given by

$$C = \max_{p_x(X)} \; I_{m|w}\left(x, y_{m|w}\right)$$

$$\text{subject to } E[x^2] \leq P$$

- We again consider the channel model described by

$$y_{m|w} \; = x + n_{m|w},$$

where $n_{m|w}$ is AWGN with zero mean and variance $\sigma_{m|w}^2$.

- We then work out the mutual information expression as follows:

$$I\left(x, y_{m|w}\right) = h\left(y_{m|w}\right) - h(y_{m|w}|x)$$

- The mutual information expression for the main channel is described by

$$
\begin{aligned}
I(x, y_m) &= h(y_m) - h(y_m|x) \\
&= h(y_m) - h(x + n_m|x) \\
&= h(y_m) - h(n_m),
\end{aligned}
$$

which takes into account that x and n_{m} are statistically independent.

- The differential entropy of the AWGN noise is given by

$$h(n_m) = \frac{1}{2}\log_2(2\pi e\sigma_m^2)$$

- The variance of y_m is given by

$$\sigma_{y_m}^2 = E[y_m^2]$$
$$= E[(x+n_m)^2] = E[x^2] + E[n_m^2] = P + \sigma_m^2$$

- The differential entropy of y_m is expressed by

$$h(y_m) = \frac{1}{2}\log_2\left(2\pi e \sigma_{y_m}^2\right)$$
$$= \tfrac{1}{2}\log_2\left(2\pi e(P+\sigma_m^2)\right)$$

- The capacity of the main channel is the maximum of the mutual information subject to the power constraint, which is

$$\mathrm{C_m} = \max I(x, y_m) = h(y_m) - h(n_m) = \tfrac{1}{2}\log_2\left(2\pi e(P+\sigma_m^2)\right) - \tfrac{1}{2}\log_2(2\pi e\sigma_m^2)$$
$$= \tfrac{1}{2}\log_2\left(1+\tfrac{P}{\sigma_m^2}\right) \text{ bits / transmission}$$

- The mutual information expression for the wiretap channel is

$$\begin{aligned} I(x, y_w) &= h(y_w) - h(y_w|x) \\ &= h(y_w) - h(x + n_w|x) \\ &= h(y_w) - h(n_w), \end{aligned}$$

which takes into account that x and n_{w} are statistically independent.

- The differential entropy of AWGN of the wiretap channel is given by

$$h(n_w) = \frac{1}{2}\log_2(2\pi e\sigma_w^2)$$

- The variance of y_w is given by

$$\begin{aligned}\sigma_{y_w}^2 &= E[y_w^2] \\ &= E[(x+n_w)^2] = E[x^2] + E[n_w^2] = P + \sigma_w^2\end{aligned}$$

- The differential entropy of y_w is expressed by

$$\begin{aligned}h(y_w) &= \frac{1}{2}\log_2\left(2\pi e \sigma_{y_w}^2\right) \\ &= \tfrac{1}{2}\log_2\left(2\pi e(P+\sigma_w^2)\right)\end{aligned}$$

- The capacity of the wiretap channel is the maximum of the mutual information subject to the power constraint, which is

$$\begin{aligned}\mathrm{C_w} &= \max I(x, y_w) = h(y_w) - h(n_w) = \tfrac{1}{2}\log_2\left(2\pi e(P+\sigma_w^2)\right) - \tfrac{1}{2}\log_2(2\pi e \sigma_w^2) \\ &= \tfrac{1}{2}\log_2\left(1 + \tfrac{P}{\sigma_w^2}\right) \text{ bits / transmission}\end{aligned}$$

- The secrecy capacity is the difference between the capacity of the main and the wiretap channels:

$$\begin{aligned} C_s &= C_m - C_w \\ &= \frac{1}{2}\log_2\left(1+\frac{P}{\sigma_m^2}\right) - \frac{1}{2}\log_2\left(1+\frac{P}{\sigma_w^2}\right) \end{aligned}$$

- Consequently, confidential communication is not possible unless the main channel has a better signal-to-noise ratio than the wiretap channel, i.e.,

$$SNR_m = \frac{P}{\sigma_m^2} > SNR_w = \frac{P}{\sigma_w^2}$$